Appendix 6: Scapula anatomical CS does not affect true GH axial rotation

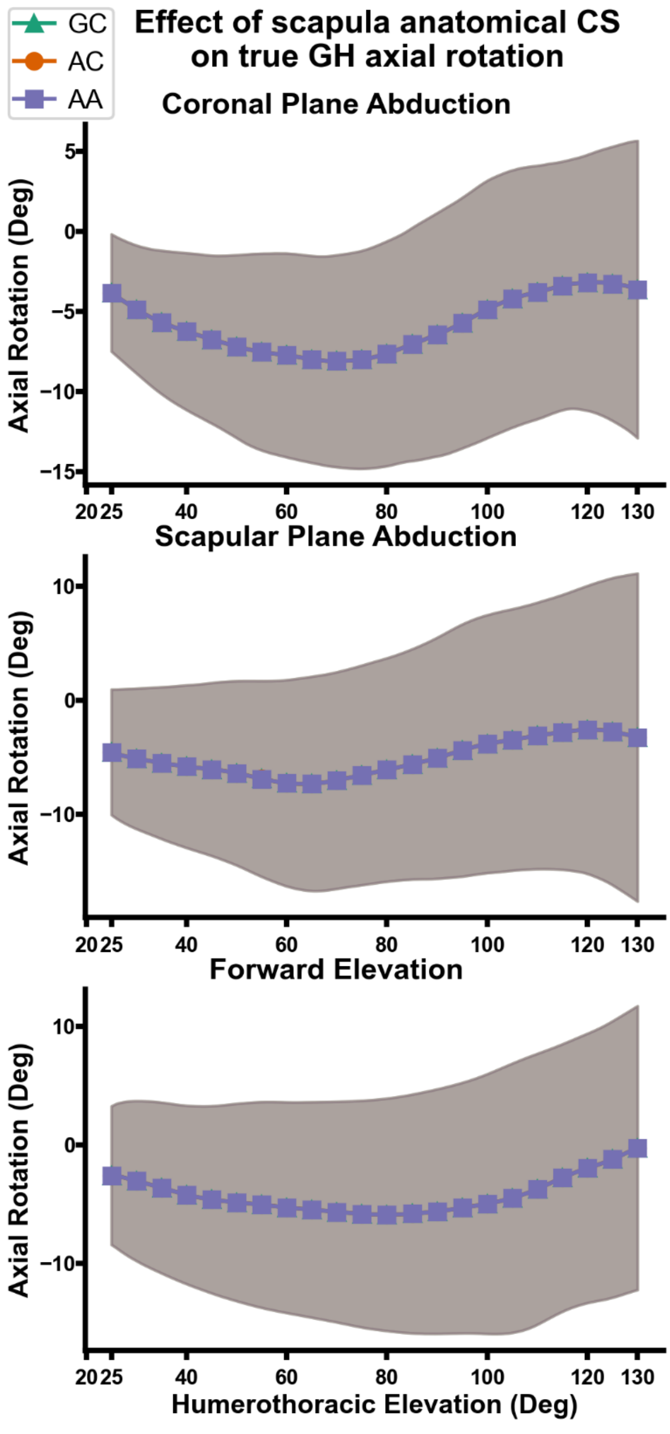

Fig A6: Effect of scapula anatomical coordinate system (CS) on true GH axial rotation. It can be shown mathematically that the scapula anatomical CS will not affect true axial rotation since it is a scalar (zero-order tensor) quantity. To demonstrate this fact numerically, true GH axial rotation was computed when using the glenoid center (GC), AC joint (AC) and the acromial angle (AA) to define the lateral direction of the scapula. The shaded region denotes ±1 SD for the coronal plane abduction (top), scapular plane abduction (middle), and forward elevation (bottom). Because the computed true axial rotation trajectories are identical, only the AA plot can be seen.

Mathematical proof of the fact that true axial rotation is a zero-order tensor follows on the next page.

Appendix 6: Scapula anatomical CS does not affect true GH axial rotation

Let $\theta(t_k)$ represent true axial rotation at time t_k, ${}^{S}\omega(t_k)$ represent GH angular velocity in the scapula's frame and ${}^{S}\mathbf{l}(t_k)$ represent the humerus' longitudinal axis in the scapula's frame. True axial rotation is computed as:

$$\theta(t_k) = \int_0^{t_k} {}^{S}\omega \cdot {}^{S}\mathbf{l}\, dt \tag{1}$$

Because we want to switch between different coordinate systems of the scapula, we will indicate these with a numbered subscript next to the S which denotes the scapula's frame. In addition, note that the right-hand side of Equation (1) can be written in matrix notation as in Equation (2):

$$\int_0^{t_k} {}^{S_1}\omega \cdot {}^{S_1}\mathbf{l}\, dt = \int_0^{t_k} {}^{S_1}\omega^T \; {}^{S_1}\mathbf{l}\, dt \tag{2}$$

Let ${}^{S_2}\mathbf{R}_{S_1}$ be an orthogonal matrix which transforms between two definitions of the scapula's frame. We show that when this change-of-basis matrix is applied to both the angular velocity vector and the longitudinal axis of the humerus, the quantity in Equation (2) remains invariant. Since, by definition, a tensor is a quantity that is invariant under a change of basis [1], this proves that true axial rotation is a zero-order tensor.

$$\begin{aligned}
\int_0^{t_k} {}^{S_1}\omega^T \; {}^{S_1}\mathbf{l}\, dt &\overset{(1)}{=} \int_0^{t_k} ({}^{S_2}\mathbf{R}_{S_1}\, {}^{S_1}\omega)^T \; {}^{S_2}\mathbf{R}_{S_1}\, {}^{S_1}\mathbf{l}\, dt \\
&\overset{(2)}{=} \int_0^{t_k} {}^{S_1}\omega^T \; {}^{S_2}\mathbf{R}_{S_1}^T \; {}^{S_2}\mathbf{R}_{S_1}\, {}^{S_1}\mathbf{l}\, dt \\
&\overset{(3)}{=} \int_0^{t_k} {}^{S_1}\omega^T \; \mathbf{I}\; {}^{S_1}\mathbf{l}\, dt \\
&\overset{(4)}{=} \int_0^{t_k} {}^{S_1}\omega^T \; {}^{S_1}\mathbf{l}\, dt
\end{aligned} \tag{3}$$

Step (2) → (3) follows from the fact that ${}^{S_2}\mathbf{R}_{S_1}$ is an orthogonal matrix.

[1] A.J.M. Spencer, Continuum mechanics, Courier Corporation2004.